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QUANTUM MECHANICAL THEORY OF NON-LINEAR  
PLASMA PHENOMENA

BY



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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled QUANTUM MECHANICAL THEORY OF NON-LINEAR PLASMA PHENOMENA, submitted by ABDEL-FATTAH ABDEL-LATIF SELIM in partial fulfillment of the requirements for the degree of Master of Science.



## ABSTRACT

An important consequence of the quasi-linear theory of plasmas is that a weakly turbulent plasma may be regarded as a gas of quasi-particles such as plasmons and photons undergoing collective oscillations and interacting with particles. In fact this process may be thought of as an emission and absorption of plasmons by particles. The wave-wave interaction involves only interactions of quasi-particle among themselves. Some non-linear plasma phenomena have been studied by starting with the Hamiltonian and making use of the second quantization formalism.

In the second chapter a set of new quasi-linear kinetic equations have been derived. The third chapter contains the study of wave-wave interaction and the last chapter deals with the effect of wave-wave interaction on the quasi-linear theory.



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C H A P T E R    1



## A BRIEF REVIEW OF THE NON-LINEAR PLASMA THEORIES

### INTRODUCTION

Preliminary studies in the theory of non-linear plasma have been made by Drummond and Pines [1] and by Vedenov, Velikov, and Sagdeev [2]. Their work has been described as a quasi-linear theory. An important deduction from this work is that certain unstable modes persist for a considerable length of time. In the theory the particle-distribution function is split into two parts - a slowly varying part called the background, and a rapidly varying part which averages out to zero. This rapidly varying part represents a system of oscillations with randomly distributed phases, whereas the slowly varying part when combined with the linear terms in the collisionless Boltzmann equation yields what is called a quasi-linear equation. However, when the non-linear terms are combined with the rapidly varying part, there results coupling between various modes. By averaging over random phases, the equation for the slowly varying distribution is obtained in the form of a diffusion equation. This diffusion establishes an equilibrium while mode-mode coupling causes damping of the spectrum.

A further deduction from the quasi-linear theory [1,3] is that a weakly turbulent plasma may be regarded as a gas of quasi-particles such as plasmons and photons undergoing



collective oscillations and interacting with the particles. In fact this process may be thought of as an emission and absorption of plasmons by particles. The mode-mode interaction involves only interactions of quasi-particles among themselves. Working in the Hamiltonian formalism instead of the Boltzmann equation, permits the use of powerful field theoretic methods which yield the additional spontaneous emission term in the kinetic equations.

Classically one has to solve coupled equations for one and two particle distribution functions in order to account for this very term which is essential for an approach to equilibrium [4].

Since the quasi-linear theory deals only with the interaction between particles and waves, there exists a possibility for including in the theory the interaction among the waves. Some estimates of these effects have already been made [1,6] for the potential waves, but these are based on the Boltzmann equation formalism. Asymptotic perturbation theories as applied to hydrodynamic equations have also been used [7,8].

The study of non-linear plasma theories is necessary for the understanding of plasma instabilities, since oscillations which are unstable in the linear approximation, may be stable in the non-linear case. Langmuir waves are most effectively excited in a plasma under certain conditions by



an impinging beam of charged particles. The non-linear interactions convert the Langmuir into transverse oscillations, which may be experimentally observed. As a rule, the generated oscillations have phase velocities of the order of the beam velocity. If the oscillations can be diverted by non-linear effects from that region of the phase space in which they can be generated by the beam, with sufficiently high efficiency, then cascade-like development of the stimulated emission process ceases and the instability is stabilized. This example further illustrates the general premise that only the investigation of non-linear effects can determine the practical significance of various plasma instabilities.

The non-linear wave interaction that leads to the Langmuir oscillations from other types of waves is therefore of interest for plasma confinement problems.

Another example of a non-linear interaction occurs in the earth's radiation belts. In the outer radiation belt electrons with energies of  $5 \times 10^4$  -  $2 \times 10^6$  ev have a measured density between  $10^{-2}$  and  $3 \times 10^{-6}$   $\text{cm}^{-3}$ . This density is far higher than can be expected in a Maxwellian distribution of a cold plasma in which the average particle energy is 1 ev and the number density is  $10^3/\text{cm}^3$ . This phenomenon can be explained if one accepts the consequences of the quasi-linear theory which predicts the possibility of a statistical acceleration of particles by a wave.



In the succeeding chapters quantum mechanical theory has been applied to investigate wave-particle and wave-wave interactions. The quasi-linear equations thus obtained agree with the previously known work [1,2]. The plasma model considered neglects the collisions between particles, and no external magnetic field is supposed to be present. The quasi-linear equations have been solved analytically and exact numerical results for the steady state have been obtained. These agree well with the results obtained numerically by Drummond and Pines [1]. In the last chapter the effect of transverse modes on quasi-linear theory has been studied.



C H A P T E R    2



QUASI-LINEAR PLASMAS THEORY

A plasma can be described as a spectrum of collective oscillations or plasmons. The frequency and the phase velocity of propagation of the plasma waves are determined by the dispersion equation which contains the macroscopic parameters of the plasma such as temperature, density, external magnetic field, etc. This is the basis of the concept that all the particles comprising the plasma partake in the plasma waves. In such a situation the dynamical aspect of the particles can be described by fluid-dynamical equations. However, this being a collective behaviour description, the phase velocity is appreciably larger than the individual particle velocities. In fact one is implicitly assuming so, when using the fluid dynamical equations.

When one examines the damping or growth of waves which is determined by the "fine structure" of the particle-distribution in phase space, the role played by the resonance particles becomes important. The resonant particles are described by

$$\omega_{\vec{k}} - \vec{k} \cdot \vec{v} = n \omega_H \quad n = 0, 1, 2, \dots \quad (2.1)$$

where  $\omega_{\vec{k}}$  and  $\vec{k}$  are the frequency and the wave vectors characterizing the wave,  $\vec{v}$  is the particle velocity, and  $\omega_H$  "the cyclotron frequency"  $= \frac{eH}{mc}$ . ( $H$  = external magnetic field,  $m$  = mass of particle). Such particles are capable of



exchanging energy with the wave and can thus damp or excite it. We shall only be concerned with the situation when  $n = 0$  or  $H = 0$  (no external magnetic field). The damping of the plasma in thermodynamic equilibrium is found to be dependent on first derivative of the particle distribution function  $f(v)$ ,  $v = \omega/k$ .

A linear theory is unsatisfactory since the growth in amplitude of a wave is unbounded. A quasi-linear theory such as that of Drummond and Pines [1], predicts a process such as that in figure (1) in which a particle and wave interact giving rise to a one particle final state.

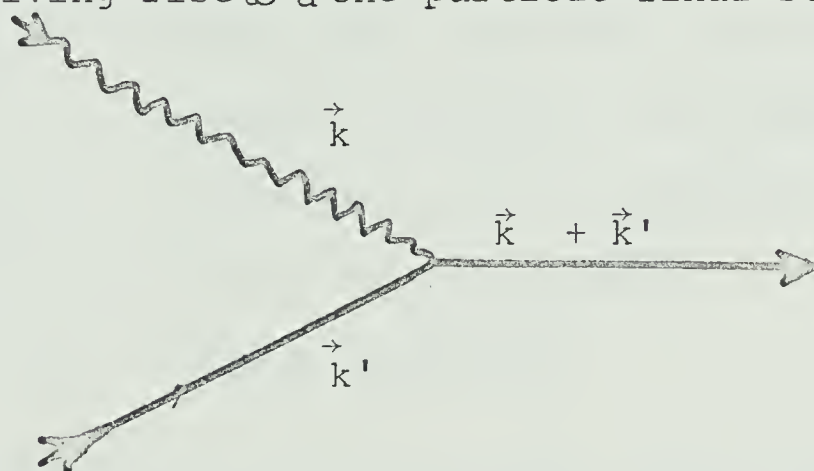


Fig. (1)

However the following processes are also possible

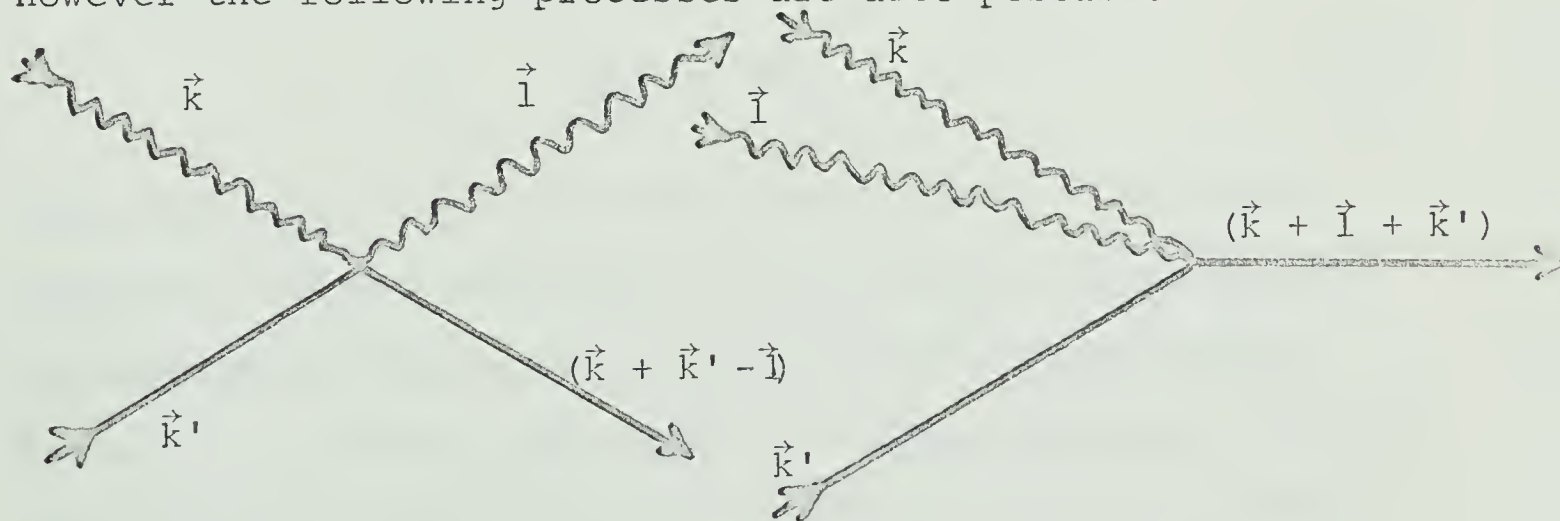


Fig. (2)

Fig. (3)



## 1. Particle-wave Formation

For an electron plasma with zero external magnetic field, the Hamiltonian  $H$  is given by

$$H = \frac{1}{2m} \sum_i (\vec{p}_i - \frac{e}{c} \vec{A}(\vec{x}_i))^2 + \frac{1}{8\pi} \int (E_T^2 + H^2) d\vec{x} + \frac{1}{2} \sum_{i < j} \frac{e^2}{|\vec{x}_i - \vec{x}_j|} \quad (2.2)$$

where

- $m$  = mass of the electron
- $\vec{p}_i$  = canonical momentum
- $\vec{E}_T$  = electrical field for the transverse waves
- $\vec{H}$  = magnetic field for the transverse waves
- $\vec{x}_i$  = coordinate of the particle
- $-e$  = charge of the electron
- and  $\vec{A}$  = vector potential

The summation is over the particles and the gauge chosen is the one in which the scalar potential is zero. The vector potential  $\vec{A}$  can be split into two parts

$$\vec{A} = \vec{A}_L + \vec{A}_T \quad (2.3)$$

such that  $\vec{A}_L$  is derivable from the gradient of a scalar function, and the divergence of  $\vec{A}_T$  vanishes. Thus  $\vec{A}_L$  can be regarded as the vector potential due to longitudinal waves and  $\vec{A}_T$  due to the transverse waves. However, since in this chapter we shall only be concerned with quasi-linear



theory, i.e. particles interacting with the longitudinal wave, we shall neglect  $\vec{A}_T$  for now, and deal with a non vanishing  $\vec{A}_T$  in the work of the next chapter. Let us make the Fourier expansion of  $\vec{A}$ :

$$\vec{A}(\vec{x}, t) = \sum_{\vec{k}} \vec{A}(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}} \quad (2.4)$$

since  $\vec{A}(\vec{x}, t)$  is real:

$$\vec{A}(-\vec{k}, t) = \vec{A}^*(\vec{k}, t) \quad (2.4a)$$

For  $k < k_D$  and  $k > k_D$ , the Hamiltonian is separable into two parts as shown by Bohm [31], where  $k_D$  is the Debye wave vector.

$$\begin{aligned} H = & \frac{1}{2m} \sum_i \vec{p}_i^2 + V \sum_{\substack{\vec{k} \\ k < k_D}} \frac{1}{8\pi c^2} \frac{\partial \vec{A}(\vec{k}, t)}{\partial t} \cdot \frac{\partial \vec{A}^*(\vec{k}, t)}{\partial t} + \\ & \frac{\omega_p^2}{8\pi c^2} \vec{A}(\vec{k}, t) \cdot \vec{A}^*(\vec{k}, t) - \frac{e}{2mc} \sum_{\substack{i, k \\ k < k_D}} [\vec{p}_i \cdot \vec{A}(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}_i} + \\ & + \vec{A}(\vec{k}, t) \cdot \vec{p}_i e^{i\vec{k} \cdot \vec{x}_i}] + \frac{1}{2} \sum_{\substack{i < j \\ k > k_D}} \frac{e^2}{|\vec{x}_i - \vec{x}_j|} \end{aligned} \quad (2.5)$$

where  $V$  is the volume,  $\vec{A}$  stands for  $\vec{A}_L$ . The second term is the Hamiltonian of a harmonic oscillator with coordinates and momenta  $\vec{A}(\vec{k}, t)$  and  $\frac{V}{4\pi c^2} \frac{\partial \vec{A}^*}{\partial t}$ . The frequency of such



oscillators is evidently  $\omega_p$ . The third term is due to the interaction between the particle and the wave, and the fourth is due to particle-particle effects. For such oscillators we can define according to Davydov [33] the normal coordinates given by

$$a = \sqrt{\frac{\mu \omega_p}{2\hbar}} \vec{A}(\vec{k}, t) + \sqrt{\frac{i}{2\mu\hbar\omega_p}} \vec{\Pi}(\vec{k}, t) \quad (2.6)$$

$$a^* = \sqrt{\frac{\mu \omega_p}{2\hbar}} \vec{A}^*(\vec{k}, t) - \sqrt{\frac{i}{2\mu\hbar\omega_p}} \vec{\Pi}^*(\vec{k}, t) \quad (2.7)$$

where  $\mu = \frac{V}{4\pi C^2}$ , and  $\vec{\Pi}(\vec{k}, t)$  is the field momenta defined above. It is easily shown that the Poisson's bracket

$$\{a(k), a^*(k')\} = -\frac{i}{\hbar} \delta_{\vec{k}\vec{k}'} \quad (2.8)$$

and the equations of motion follow from

$$\frac{\partial a(k)}{\partial t} = \{a(k), H\} = -\frac{\partial H}{\partial a(k)} \quad (2.9)$$

The transition to quantum mechanics is now made as follows

$$\begin{aligned} \{a(k), a^*(k')\} \Rightarrow -\frac{i}{\hbar} \{a(k), a^*(k')\} \Rightarrow -\frac{i}{\hbar} \left( a(k) \right. \\ \left. a^*(k') - a^*(k') a(k) \right) = -\frac{i}{\hbar} \delta_{\vec{k}\vec{k}'} \end{aligned} \quad (2.10)$$



so that, the commutation relation is

$$[a(k), a^*(k')] = \delta_{\vec{k}\vec{k}'} \quad (2.11)$$

Thus in the transition to quantum mechanics  $a(k)$  and  $a^+(k)$  become the second quantized operators for plasmons of momentum  $\hbar\vec{k}$ . If  $|N(\vec{k})\rangle$  is a state vector with  $N(\vec{k})$  plasmons in the state  $\vec{k}$ , it follows

$$a^+(\vec{k}) | \dots N(\vec{k}) \dots \rangle = \sqrt{N(\vec{k})+1} | \dots N(\vec{k})+1 \dots \rangle \quad (2.12)$$

$$a(\vec{k}) | \dots N(\vec{k}) \dots \rangle = \sqrt{N(\vec{k})} | \dots N(\vec{k})-1 \dots \rangle \quad (2.13)$$

The operators  $a^+(k)$  and  $a(k)$  create and destroy respectively one plasmon from a state  $\vec{k}$ .

The second quantization Hamiltonian for the wave-particle interaction can simply be written by using (2.6) and (2.7) and

$$H_{\omega-p} = - \frac{e}{2mc} \int d\vec{x} \psi^+(x, t) \sum_{i,k} [\vec{p}_i \cdot \vec{A}(k, t) + \vec{A}(\vec{k}, t) \cdot \vec{p}_i] e^{i\vec{k} \cdot \vec{x}} \psi(x, t) \quad (2.14)$$

where

$$\psi(\vec{x}, t) = \sum_k b(k) \xi(\vec{x}) \quad (2.15)$$

$$\psi^+(\vec{x}, t) = \sum_k b^+(k) \xi^+(\vec{x}) \quad (2.16)$$



where  $\xi(\vec{x})$  is the plane wave function for the particle,  $b^+$  and  $b$  are the creation and annihilation operators for the particles.

After using (2.6) to (2.16), the Hamiltonian finally becomes (after neglecting particle-particle interactions.)

$$H = H_0 + H_{\text{int}} \quad (2.17)$$

with

$$H_0 = \frac{1}{2} \hbar \sum_{\vec{k}} \omega_p (a a^+ + a^+ a) \quad (2.18)$$

$$H_{\text{int}} = \sum_{\vec{k}, \vec{k}'} \frac{e\hbar}{mc} \kappa' \sqrt{\frac{\hbar}{2\mu\omega_p}} \left[ a(\vec{k}) b^+(\vec{k}') b(\vec{k}' - \vec{k}) + a^+(\vec{k}) b^+(\vec{k}' - \vec{k}) b(\vec{k}') \right] + \text{H.C.} \quad (2.19)$$

$H_{\text{int}}$  will give us the probabilities for emission and absorption of plasmons by the particles.

## 2. The Development of the Distribution

If a larger number of plasmons are emitted in a state than are absorbed from it, the wave is unstable. The wave is stable if the reverse happens. We have to calculate the net number of plasmons in a state. If  $N(\vec{k})$  and  $f(\vec{k})$  are the



number of plasmons and particles in a state  $k$ , then,  
schematically

$$\frac{dN(\vec{k})}{dt} = \sum_{\vec{k}'} \left( \begin{array}{c} \vec{k} + \vec{k}' \\ \text{---} \end{array} \right) \begin{array}{c} \vec{k} \\ \text{---} \end{array} \begin{array}{c} \vec{k} \\ \text{---} \end{array} \begin{array}{c} \vec{k}' \\ \text{---} \end{array} \begin{array}{c} \vec{k}' \\ \text{---} \end{array} \right) - \left( \begin{array}{c} \vec{k}' \\ \text{---} \end{array} \begin{array}{c} \vec{k}' \\ \text{---} \end{array} \begin{array}{c} \vec{k} \\ \text{---} \end{array} \begin{array}{c} \vec{k} + \vec{k}' \\ \text{---} \end{array} \right) \quad (2.20)$$

or

$$\frac{dN(\vec{k})}{dt} = \frac{2\pi}{\hbar} \sum_{\vec{k}'} |M|^2 \left[ f(\vec{k}' + \vec{k}) - f(\vec{k}') \right] N(\vec{k}) \delta(\hbar\omega_p + E_{k'} - E_{k+k'}) \quad (2.20a)$$

where  $M$  is the vertex part and  $E$ 's are the particle energies.

Similarly

$$\begin{aligned} \frac{df(\vec{k}')}{dt} &= \frac{2\pi}{\hbar^2} \sum_{\vec{k}} |M|^2 N(\vec{k}) \left( \frac{\hbar}{m} \right)^2 \left( \frac{m}{\hbar} \right)^2 k^2 \frac{\partial}{\partial k'} \\ &\left[ \delta\left(\omega_p - \frac{\hbar \vec{k} \cdot \vec{k}'}{m}\right) \times \frac{\partial f(k')}{\partial k'} \right] + \\ &\frac{2\pi}{\hbar^2} \sum_{\vec{k}} |M|^2 \vec{k} \cdot \frac{\partial}{\partial k'} \left[ \delta\left(\omega_p - \frac{\vec{k} \cdot \hbar \vec{k}'}{m}\right) f(k') \right] \end{aligned} \quad (2.21)$$



The second term above is the spontaneous emission term, which is absent in the classical theories.

If we take the expectation value of  $|A(k)|^2$  between states that are eigen functions of the number operator, then

$$|A(k)|^2 = \frac{\chi_N(k)}{\mu \omega_p}$$

$$\text{Therefore } \mathcal{E} = \frac{1}{c^2} \frac{\omega_p^2}{8\pi} |A(k)|^2 = \frac{1}{c^2} \frac{\omega_p^2}{8\pi} \frac{\chi_N(k)}{\mu \omega_p} \quad (2.22)$$

where  $\mathcal{E}$  is the electric energy density in the waves.

The classical limit of (2.20) and (2.20a) give the fundamental equations of quasi-linear theory. If one neglects the small spontaneous emission term, then these equations are

$$\frac{\partial \mathcal{E}}{\partial t} = \alpha(v) \mathcal{E}(v, t) - \frac{\partial f(v)}{\partial v} \quad (2.23)$$

$$\frac{\partial f(v)}{\partial t} = \frac{\partial}{\partial v} \left( \beta(v) \mathcal{E}(v, t) - \frac{\partial f(v)}{\partial v} \right) \quad (2.24)$$

$$\text{where } \alpha(v) = \frac{4\pi^2 e^2 v^2}{m \omega_p^2}, \text{ and } \beta(v) = \frac{8\pi L e^2}{m^2 v}$$

where  $L$  is the side of the cube containing the plasma.

These equations are identical to those obtained in [1,2].

By some algebraic manipulation these equations yield

$$\frac{\partial}{\partial t} \left[ \frac{1}{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial t} - \alpha(v) \frac{\partial^2}{\partial v^2} \left( \frac{\beta}{\alpha} \mathcal{E} \right) \right] = 0 \quad (2.24a)$$



Therefore

$$\frac{1}{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial t} - \alpha(v) \frac{\partial^2}{\partial v^2} \left( \frac{\beta}{\alpha} \mathcal{E} \right) = \text{constant} \quad (2.25)$$

at  $t = 0$ ,  $f = F_0$ , where  $F_0$  is given by [1]

$$F_0 = \frac{n}{(2\pi)^{1/2} \bar{v}} \text{Exp} \left[ -\frac{1}{2} \left( \frac{v}{\bar{v}} \right)^2 \right] + 4 \times 10^{-4} \text{Exp} \left[ -\frac{1}{2} \left( \frac{v}{\bar{v}} - 5 \right)^2 \right] \quad (2.26)$$

and  $\mathcal{E} = \mathcal{E}_0$  then

$$\frac{1}{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial t} - \alpha(v) \frac{\partial^2}{\partial v^2} \left( \frac{\beta}{\alpha} \mathcal{E} \right) = \alpha(v) \left[ \frac{\partial}{\partial v} F_0 \left( \frac{v}{\bar{v}} \right) - \frac{\partial^2}{\partial v^2} \left( \frac{\beta}{\alpha} \mathcal{E}_0 \right) \right] \quad (2.27)$$

In the steady state  $\mathcal{E}$  is described by the equation

$$\mathcal{E} - \mathcal{E}_0 = \frac{\alpha}{\beta} \left[ \int_0^\infty F_0 \left( \frac{v}{\bar{v}} \right) dv - \int_0^v F_0 \left( \frac{v}{\bar{v}} \right) dv \right] \quad (2.28)$$

or

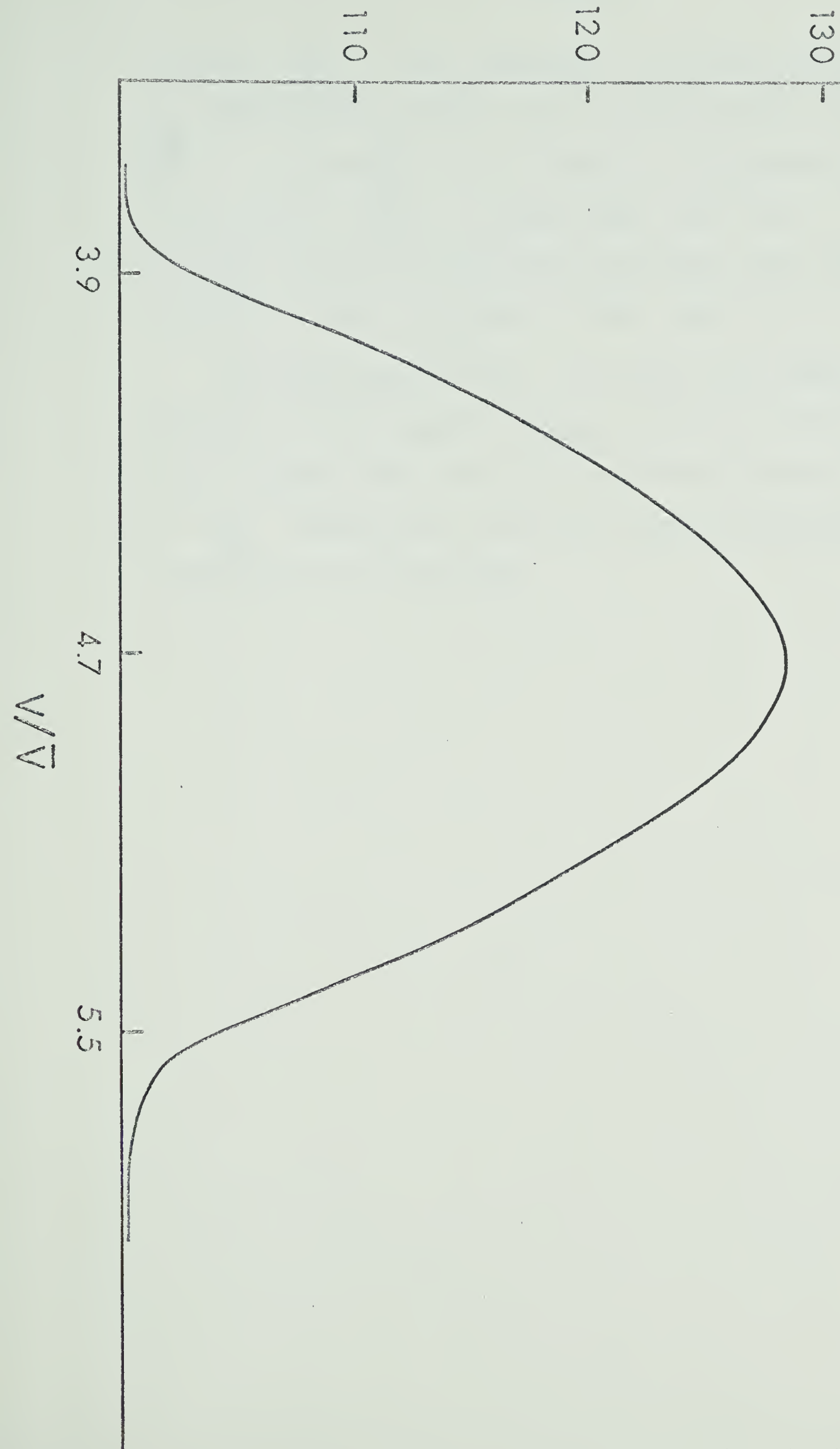
$$\mathcal{E}' = \left( \frac{v}{\bar{v}} \right)^3 \left[ \int_0^\infty F_0 \left( \frac{v}{\bar{v}} \right) dv - \int_0^v F_0 \left( \frac{v}{\bar{v}} \right) dv \right] \quad (2.28a)$$

where  $\mathcal{E}' = \frac{2L^{\omega} p}{\bar{v}^2 \pi m} (\mathcal{E} - \mathcal{E}_0)$

The integrals of the type  $\int_0^v F_0 \left( \frac{v}{\bar{v}} \right) dv$  can only be solved numerically [19, 15]. The fig. (4) shows  $\mathcal{E}'$



$$\varepsilon' (V/\bar{V})$$





against  $(\frac{v}{v})$ . This determines the steady state spectrum of electrostatic energy. It is worth noting that when  $\frac{\partial \mathcal{E}}{\partial t} = 0$ , then  $\frac{\partial f}{\partial v} = 0$ . Thus the distribution becomes flat near  $v = \omega_p/k$  at very large time  $\omega_p t \rightarrow \infty$ . The curve in fig. (4) is in fairly good agreement with the previous work [1, 2, 3] for values of the order of  $t \sim \frac{10,000}{\omega_p}$  sec.

The spectrum  $\mathcal{E}(\frac{\omega_p}{k})$  that has been obtained by the methods of this chapter produce a particle distribution which differs from the Maxwellian distribution only for  $v \sim \frac{\omega_p}{k}$ . Details of the other interactions mentioned are contained in the Appendix.



C H A P T E R    3



COUPLING OF THE LONGITUDINAL TO TRANSVERSE WAVES

While early studies of non-linear wave-wave interactions were carried out by Ginzburg [16] and Bailey [17] it is only more recently that the theory has been developed to such an extent that the physical mechanisms are now well understood. The rapidly growing experimental research in this area of research has provided an added incentive for the pursuit of theoretical studies. To date attention has been directed mainly to cases of weak non-linearity wherein the wave-wave modes which result from the weak non-linear interaction can develop to a significant wave-amplitude as the system approaches equilibrium.

The first study of non-linear interaction specifically with respect to Langmuir oscillations was due to Sturrock [18]. Further development of the theory was made by Drummond and Pines [1], and Zheleznyakov [19]. The first attempt to study the non-linear effects in the framework of kinetic approach was made by Kodomtsev and Petviashvili [20], Karpman [21], Galeev, Karpman, and Sagdeev [22], Camoc [23] and Rudakov, Vedenov et al [24]. Some non-linear problems connected with Langmuir waves were also studied by Silin [25] who used the correlation method developed by Bogolyubov [34]. This approach is mathematically so involved that it does not permit extensive use in some difficult problems.



In this chapter only the transverse-longitudinal wave-wave interaction will be considered in detail. The problem of excitation of transverse plasma modes [27,29] and the energy transferred to them [26] by non-linear interaction of longitudinal waves has previously been studied.

### 1. Wave-Interaction Formalism

For an electron plasma without any external field, the Hamiltonian  $H$  is the same as given by (2.2) together with the definition (2.3)

$$H = \frac{1}{2m} \sum_i \left( \vec{p}_i - \frac{e}{c} \vec{A}(\vec{x}_i) \right)^2 + \frac{1}{8\pi} \int (\vec{E}_T^2 + \vec{H}^2) d\vec{x} + \frac{1}{2} \sum_{i < j} \frac{e^2}{|\vec{x}_i - \vec{x}_j|} \quad (3.1)$$

The Hamiltonian (3.1) can be separated into two parts, the collective and the particle.

$$H = \sum_i \frac{e^2}{2mc^2} \vec{A}^2(\vec{x}_i) + \frac{1}{8\pi} \int (\vec{E}_T^2 + \vec{H}^2) d\vec{x} + \frac{1}{8\pi} \int E_L^2 d\vec{x} \quad (3.2)$$

where  $E_L$  is the longitudinal electric field.



In (3.2) the  $(\vec{A} \cdot \vec{p})$  term which is responsible for particle-wave interaction is absent unlike the case of second chapter. This is so because for large wavelength plasma oscillations only a few particles are resonantly coupled with the wave. However, in the Hamiltonian the contribution due to transverse waves, which was neglected in the formulation of Chapter 2, is now a significant factor. Expanding the vector potential as before in a cubic box of length  $l$

$$\vec{A}_{L,T}(x,t) = \sum_{\vec{k}} \vec{A}_{L,T}(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}}$$

with

$$\vec{A}_{L,T}^*(\vec{k}, t) = \vec{A}_{L,T}(-\vec{k}, t) \quad (3.4)$$

Using the Poisson's equation, the Hamiltonian becomes

$$H = H_T + H_L + H_{int} , \quad (3.5)$$

where

$$H_T = V \sum_{\vec{k}} \left( \frac{1}{8\pi c^2} \frac{\partial \vec{A}_T}{\partial t}(\vec{k}, t) \cdot \frac{\partial \vec{A}_T^*}{\partial t}(\vec{k}, t) + \frac{k^2}{8\pi} \vec{A}_T(\vec{k}, t) \cdot \vec{A}_T^*(\vec{k}, t) + \frac{\omega_p^2}{8\pi c^2} \vec{A}_T(\vec{k}, t) \cdot \vec{A}_T^*(\vec{k}, t) \right), \quad (3.6)$$

$$H_L = V \sum_{\vec{k}} \left( \frac{1}{8\pi c^2} \frac{\partial \vec{A}_L}{\partial t}(\vec{k}, t) \cdot \frac{\partial \vec{A}_L^*}{\partial t}(\vec{k}, t) + \frac{\omega_p^2}{8\pi c^2} \vec{A}_L(\vec{k}, t) \cdot \vec{A}_L^*(\vec{k}, t) \right), \quad (3.7)$$



$$H_{\text{int}} = \frac{\omega_p e}{m 8 \pi c} \frac{1}{3} V \sum_{\vec{k}, \vec{k}'} \left[ \vec{A}_L(\vec{k}, t) \cdot \vec{A}_L(\vec{k}', t) + \vec{A}_T(\vec{k}', t) \cdot \vec{A}_L(\vec{k}, t) + \right. \\ \left. \vec{A}_L(\vec{k}, t) \cdot \vec{A}_T(\vec{k}', t) + \vec{A}_T(\vec{k}', t) \cdot \vec{A}_T(\vec{k}, t) \right] - \vec{A}_L^*(\vec{k} + \vec{k}', t) \cdot (\vec{k} + \vec{k}'), \quad (3.8)$$

and  $V$ , the volume.

$H_T$  is the Hamiltonian for the transverse waves,  $H_L$  for the longitudinal waves and  $H_{\text{int}}$  for the interaction between the two. That part of the Hamiltonian which is responsible for the self interaction of the waves has been neglected.

For the purpose of forming the plasmon Hamiltonian, it is convenient to introduce the canonical variable for the longitudinal and transverse waves,

$$\vec{\pi}_{L,T} = \frac{V}{4 \pi c^2} \frac{\partial \vec{A}}{\partial t}{}_{L,T} \quad (3.9)$$

$$\vec{\phi}_{L,T} = \vec{A}_{L,T} \quad (3.10)$$

where  $\vec{\pi}$  is the momentum and  $\vec{\phi}$  the coordinate. Next we can form the plasmon normal coordinates defined by

$$a_{L,T} = \sqrt{\frac{\mu \omega_{T,L}(k)}{2 \hbar}} \vec{A}_{L,T} + \frac{i}{\sqrt{2 \mu \hbar \omega_{T,L}(k)}} \vec{\pi}_{L,T}, \quad (3.11)$$

$$a_{L,T}^* = \sqrt{\frac{\mu \omega_{T,L}(k)}{2 \hbar}} \vec{A}_{L,T}^* - \frac{i}{\sqrt{2 \mu \hbar \omega_{T,L}(k)}} \vec{\pi}_{L,T}^* \quad (3.12)$$



where

$$\mu = \frac{v}{4\pi c^2}, \quad \omega_L^2 = \omega_p^2 \quad \text{and} \quad \omega_T^2 = (k^2 c^2 + \omega_p^2) \quad (3.13)$$

The  $a$ 's and  $a^*$ 's are the annihilation and creation operators of second quantization for the plasmons.

Using (3.9-13) we can write (3.6-8) as

$$H_T = \frac{1}{2} \hbar \sum_k \omega_T(k) (a_T a_T^* + a_T^* a_T) \quad (3.14)$$

$$H_L = \frac{1}{2} \hbar \sum_k \omega_L (a_L a_L^* + a_L^* a_L) \quad (3.15)$$

$$H_{int} = H_1 + H_2 \quad (3.16)$$

$$H_1 = \frac{\omega_{pe}}{m8\pi c^3} V \sum_{kk'} \frac{\hbar^{3/2}}{\sqrt{\mu \omega_p^2 8\omega_T(k) \mu}} (k+k') \cos\theta [a_T^*(\vec{k}) a_L^*(\vec{k}') a_L(k+k') + a_T(k) a_L^*(k') a_L(k+k') + a_T^*(k) a_L(k') a_L(k+k') + a_T(k') a_L^*(k) a_L(k+k') + H.C.] \quad (3.17)$$

$$H_2 = \frac{\omega_{pe}}{m8\pi c^3} V \sum_{kk'} \frac{\hbar^{3/2}}{\sqrt{\mu 8\omega_T(k') \omega_T(k) \omega_p^2 \mu}} K \cos\theta (a_T^*(k) a_T^*(k') a_L(k+k') + a_T(k) a_T^*(k') a_L(k+k') + a_T^*(k) a_T(k') a_L(k+k') + a_T(k) a_T(k') a_L(k+k')) + H.C., \quad (3.18)$$



$H_T$  and  $H_L$  are the oscillator energy of the transverse and longitudinal waves.  $H_1$  is due to two longitudinal waves colliding producing a single transverse wave and  $H_2$  represents two processes: (a) two transverse waves collide to produce one longitudinal wave (b) one longitudinal and one transverse wave interact to result in one transverse wave.

Thus from  $H_{int}$  we can write the matrix elements for the respective transitions. Knowing this and using the "Golden Rule" for time dependent transitions, we will be in a position to write the time development for the plasmon distribution.

## 2. Plasmon Distribution

If  $N(k)$  and  $n(k')$  are the number of plasmons for the longitudinal and transverse waves respectively and if the curly line denotes longitudinal wave and the straight line transverse wave, then schematically



$$\begin{aligned}
 \frac{dN(K)}{dt} = & \sum_{K'} \left[ \begin{array}{c} K'-K \\ \nearrow \text{wavy} \\ K' \rightarrow \text{vertex} \\ \searrow \text{wavy} \\ K \end{array} \right] - \left[ \begin{array}{c} K'-K \\ \nwarrow \text{wavy} \\ K \leftarrow \text{vertex} \\ \rightarrow \text{wavy} \\ K' \end{array} \right] \\
 & + \sum_{K'} \left[ \begin{array}{c} K-K' \\ \nwarrow \text{wavy} \\ K \leftarrow \text{vertex} \\ \rightarrow \text{wavy} \\ K' \end{array} \right] - \left[ \begin{array}{c} K-K' \\ \nearrow \text{wavy} \\ K \rightarrow \text{vertex} \\ \nwarrow \text{wavy} \\ K' \end{array} \right] \\
 & + \sum_{K'} \left[ \begin{array}{c} K-K' \\ \nwarrow \text{wavy} \\ K \leftarrow \text{vertex} \\ \rightarrow \text{wavy} \\ K' \end{array} \right] - \left[ \begin{array}{c} K-K' \\ \nearrow \text{wavy} \\ K \rightarrow \text{vertex} \\ \nwarrow \text{wavy} \\ K' \end{array} \right]
 \end{aligned}
 \tag{3.19}$$



Similarly one can write an equation for  $\frac{dn}{dt}(k')$ . So long as the longitudinal wave vector  $k_L \ll \frac{\omega_p}{c}$  the last two interactions in (3.19) do not conserve energy. Thus in this limit we are left with only the first process.  $k_L \ll \omega_p/c$  implies that we need little longer wavelengths than are necessary to satisfy  $\lambda \ll \lambda_D$ . However if the last two processes have to play any significant role, the particle-wave interactions cannot be neglected.

In the limit  $k_L \ll \omega_p/c$

$$\begin{aligned} \frac{dN(k)}{dt} = & \frac{2\pi}{\hbar} \sum_{k'} |M_1|^2 [n(\vec{k}') \{ N(\vec{k}) + N(\vec{k}' - \vec{k}) \} - \\ & N(\vec{k}) N(\vec{k}' - \vec{k})] \delta [\hbar (\omega_T(\vec{k}') - 2\omega_p)] \end{aligned} \quad (3.20)$$

$$\begin{aligned} \frac{dn(k')}{dt} = & \frac{2\pi}{\hbar} \sum_k |M_1'|^2 [N(\vec{k}) N(\vec{k}' - \vec{k}) - n(\vec{k}') \times \\ & \{ N(\vec{k}) + N(\vec{k}' - \vec{k}) \}] \delta [\hbar (\omega_T(\vec{k}') - 2\omega_p)] \end{aligned} \quad (3.21)$$

where  $M_1$  and  $M_2$  are the matrix elements for the respective transitions and can be computed simply from the Hamiltonian (3.17). If we take the expectation value of  $|A_{L,T}|^2$  between states that are eigen functions of the number operator, then

$$\langle |A_L(k)|^2 \rangle = \frac{\hbar N(k)}{\mu \omega_p} \quad (3.22)$$



$$\langle A_T(k)^2 \rangle = \frac{\hbar n(k)}{\mu \omega_T(k)} \quad (3.23)$$

for  $N(k), n(k) \gg 1$ .

We shall assume that the longitudinal spectrum [32] is zero outside the interval  $\Delta \vec{K}$  and constant within  $\Delta \vec{K}$  and of magnitude  $|E(t)|^2$ , where  $\Delta \vec{K} = \frac{2\pi}{3} \lambda_D^{-3}$ . If at particular instant the interaction comes to life and if the longitudinal spectrum is known at that time, then the transverse energy density  $W_T(t)$  can simply be written down from the equations (3.20) and (3.21).

$$W_T(t) = \frac{7}{32} \frac{\beta}{\pi \alpha} |E_0|^2 \left[ 1 - \frac{\beta}{\alpha + \beta - \alpha \exp \left[ - \frac{c^2}{2 \omega_p} |E_0|^2 \tau \beta \right]} \right] \quad (3.24)$$

where  $\tau (> \tau_c)$  is the interval of time elapsed after the longitudinal waves had started interacting to generate transverse waves,  $\tau_c$  being the characteristic time of interaction.

$$\beta = \frac{8\pi^{3/3}}{3} \frac{L^6}{2\pi} \frac{\omega_p^4}{c^7} \frac{e^2}{m^2} K_D^3 \quad \alpha = \frac{\pi^{2/3} \omega_p^4 e^2}{m^2 c^7} \frac{L^3}{2\pi}$$

$|E_0|^2 = \frac{\chi T}{2L^3}$  is the value of  $|E|^2$  in the absence of interaction,  $\chi$  being the Boltzmann constant. Therefore the characteristic time for the growth of transverse



energy is given by

$$\frac{c^2}{\omega_p^2} |E_O|^2 \beta \approx 1 \quad \text{or}$$

$$\tau \approx \frac{16\sqrt{3}\pi^5}{\omega_p \left(\frac{\delta n}{n}\right)^2 \bar{v}^5/c^5} \quad (3.25)$$

where  $\delta n$  is the change in the electron density  $n$  and  $\bar{v}$  the thermal velocity.

If one calculates the rate of growth of energy in the transverse modes for  $\omega_p t < 1$ ,

$$\frac{dW_T(t)}{dt} = 3.2 \times 10^{-5} \left(\frac{\bar{v}}{c}\right)^5 \omega_p \frac{n\chi T}{(n\lambda_D)^3} \quad (3.26)$$

This agrees with the previous work [26]. Thus the present formalism makes it possible to derive the previously known results as a special case. Moreover it throws a light on other possible third order interactions and how they can be eliminated. Another contribution of the present chapter has been the calculation of the characteristic time for the energy transfer to transverse waves.



C H A P T E R     4



EFFECT OF INTERACTION BETWEEN TRANSVERSE AND LONGITUDINAL  
WAVES ON QUASI-LINEAR THEORY

The result is developed in Chapter 2, that in the quasi-linear approximation, the plasma oscillations have a steady state particle distribution function with a plateau near  $v = \frac{\omega}{k}$ . To date no work has been done on the effect of non-linear interactions between longitudinal and transverse waves on the particle velocity distribution. However it is important to examine this problem since, for times which are sufficiently long for the steady state to be attained, wave-wave interactions become important.

By similar arguments to those which were used to derive (2.20) and (3.19), the particle-wave and wave-wave effects can be combined into one single equation. As before the interaction in which longitudinal waves decay into two transverse waves will not be considered. Therefore schematically one obtains



$$\frac{dN(K)}{dt} =$$

The equation is represented by four Feynman diagrams. The first diagram shows an incoming fermion line with momentum  $K'$  and an outgoing fermion line with momentum  $K$ , connected by a wavy line labeled  $T$ . The second diagram is similar but with different momentum assignments. The third diagram shows an incoming fermion line with momentum  $K'$  and an outgoing fermion line with momentum  $K$ , connected by a wavy line labeled  $L$ . The fourth diagram shows an incoming fermion line with momentum  $K'$  and an outgoing fermion line with momentum  $K$ , connected by a wavy line labeled  $T$ .

(4.1)



The first summation is due to the transverse-longitudinal wave-wave and the second due to wave-particle interactions. Upon applying the principle of detailed balancing for time independent transitions to (4.1) one obtains

$$\begin{aligned} \frac{dN(k)}{dt} = & \frac{2\pi}{\hbar} \sum_{\vec{k}'} |M_1|^2 [N(\vec{k}) \{n(\vec{k}') - N(\vec{k}' - \vec{k})\} + \\ & N(\vec{k}' - \vec{k}) n(\vec{k}')] \delta [\hbar(\omega_T(\vec{k}') - 2\omega_p)] \\ & + \frac{2\pi}{\hbar} \sum_{\vec{k}'} |M_2|^2 [f(\vec{k} + \vec{k}') - f(k')] N(k) \times \\ & \delta [\hbar\omega_p + E_{\vec{k}'} - E_{\vec{k} + \vec{k}'}] \end{aligned} \quad (4.2)$$

where  $n(\vec{k})$  is number of transverse plasmons or the quasi-photons. Similarly one can write for  $n(\vec{k})$

$$\begin{aligned} \frac{dn(k')}{dt} = & \frac{2\pi}{\hbar} \sum_{\vec{k}} |M_1'|^2 [N(\vec{k}' - \vec{k}) N(\vec{k}) - n(\vec{k}')] \\ & [N(\vec{k}' - \vec{k}) + N(k)] \delta [\hbar(\omega_T(\vec{k}') - 2\omega_p)] \end{aligned} \quad (4.3)$$

The equation for  $\frac{df(k')}{dt}$  is

$$\begin{aligned} \frac{df(k')}{dt} = & \frac{2\pi}{\hbar^2} \sum_{\vec{k}} |M_2|^2 N(k) \left(\frac{\hbar}{m}\right)^2 \left(\frac{m}{\hbar}\right)^2 k'^2 \frac{\partial}{\partial k'} \\ & \delta(\omega_p - \frac{\hbar \vec{k}' \cdot \vec{k}}{m}) \frac{\partial f(k')}{\partial k'} + \end{aligned}$$



$$+ \frac{2\pi}{\hbar^2} \sum_k |M_2|^2 \quad K \quad \frac{\partial}{\partial k'} \delta \left( \omega_p - \frac{\hbar \vec{k}' \cdot \vec{k}}{m} \right) f(k') \quad (4.3a)$$

After some algebra, the fundamental equations for the elementary processes become

$$\begin{aligned} \frac{d \mathcal{E}_L(k)}{dt} = & \frac{4e^2 v}{\sqrt{3} m^2 c^5} \left( ck + \sqrt{3} \frac{\omega_p}{c} \right)^2 \left[ \frac{1}{2} \mathcal{E}_T \left( \sqrt{3} \frac{\omega_p}{c} \right) \times \right. \\ & \left. \left( \mathcal{E}_L \left( \sqrt{3} \frac{\omega_p}{c} - k \right) + \mathcal{E}_L(k) \right) - \mathcal{E}_L(k) \mathcal{E}_L \left( \sqrt{3} \frac{\omega_p}{c} - k \right) \right] + \\ & \alpha(v) \frac{\partial f}{\partial v} \mathcal{E}_L(k) \end{aligned} \quad (4.4)$$

$$\begin{aligned} \frac{d \mathcal{E}_T(k')}{dt} = & \frac{4 v e^2}{\sqrt{3} m^2 c^5} \sum_k \left( ck + \sqrt{3} \frac{\omega_p}{c} \right)^2 \left[ \mathcal{E}_L(k) \mathcal{E}_L \left( \sqrt{3} \frac{\omega_p}{c} - k \right) \right. \\ & \left. - \frac{1}{2} \mathcal{E}_T \left( \sqrt{3} \frac{\omega_p}{c} \right) \left( \mathcal{E}_L(k) + \mathcal{E}_L \left( \sqrt{3} \frac{\omega_p}{c} - k \right) \right) \right] \end{aligned} \quad (4.5)$$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left( \beta(v) \mathcal{E}_L(v, t) \right) \frac{\partial f}{\partial t} \quad (4.6)$$

where  $\mathcal{E}_L$  and  $\mathcal{E}_T$  are the electric energy densities of the longitudinal and transverse and

$$v = \frac{\omega_p}{k} \quad (4.7)$$



$$\alpha(v) = \frac{4\pi^2 e^2 v^2}{m^{\omega} p} \quad (4.8)$$

$$\beta(v) = \frac{8\pi L e^2}{m^2 v} \quad (4.9)$$

In the steady state equation (4.5) yields

$$\frac{1}{2} \mathcal{E}_T = \mathcal{E}_L \left( \sqrt{3} \frac{\omega_p}{c} \right) \frac{\int (ck + \sqrt{3} \omega_p)^2 \mathcal{E}_L(k) dk}{\int (ck + \sqrt{3} \omega_p)^2 \left[ \mathcal{E}_L(k) + \mathcal{E}_L(k - \sqrt{3} \frac{\omega_p}{c}) \right] dk} \quad (4.10)$$

If  $k \gg \sqrt{3} \frac{\omega_p}{c}$  and, of course, less than  $k_D$ , then

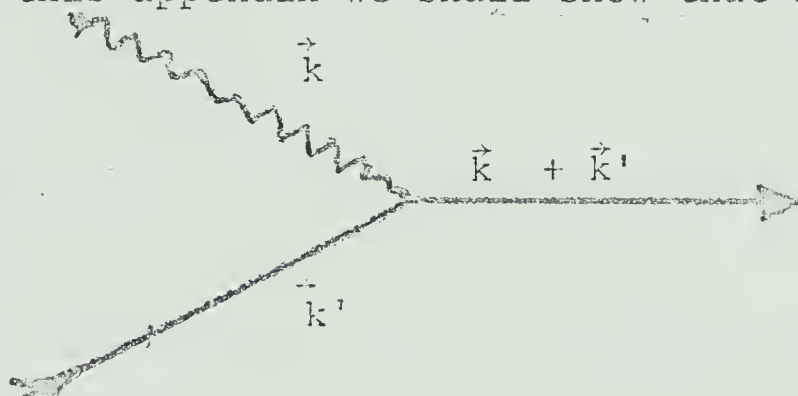
$$\mathcal{E}_T = \mathcal{E}_L \left( \sqrt{3} \frac{\omega_p}{c} \right) \quad (4.11)$$

Substituting (4.11) into (4.4), we see that the resulting equation together with (4.6) are the equations of the quasi-linear theory. Thus the transverse-longitudinal wave interaction leads to a redistribution of the energy balance in the various modes, the spectral shape of the longitudinal modes is not affected by this type of interaction. The distribution function, however, displays the same characteristic properties of Chapter 2.

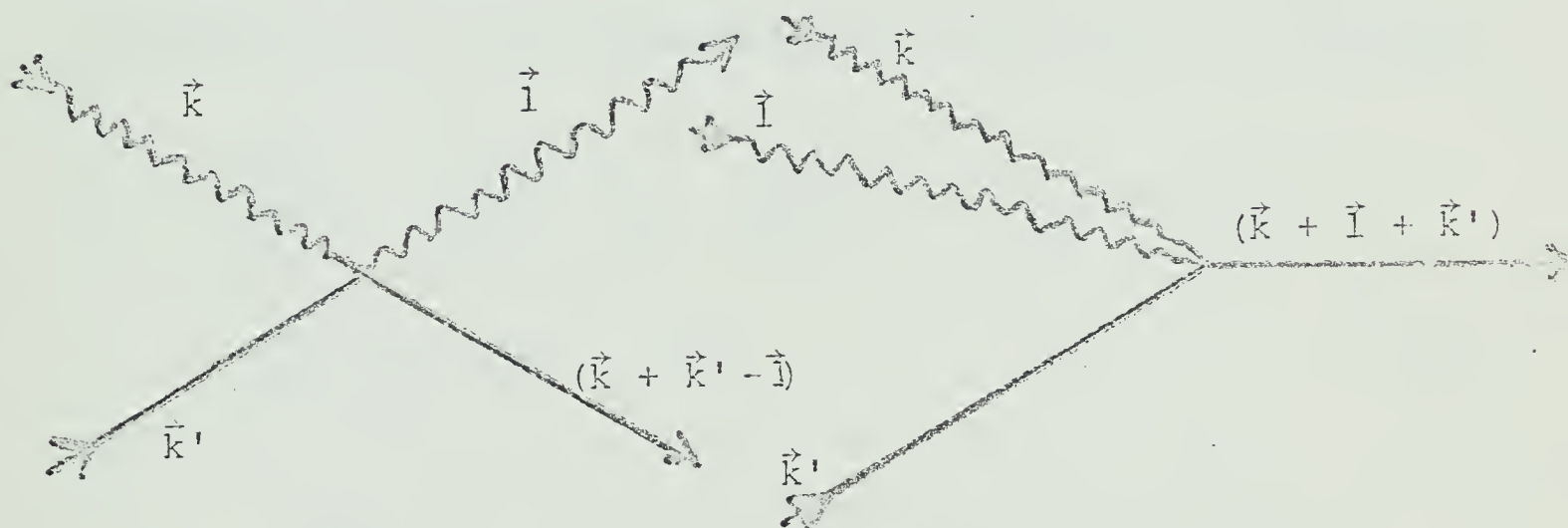


APPENDIX.

In this appendix we shall show that apart from



the following processes are also possible:



consider the term

$$H_c = \frac{1}{2m} \frac{e^2}{c^2} \sum_i A^2 (\vec{x}_i) \quad (A.1)$$

After Fourier analysing (A.1) becomes

$$H_c = \frac{1}{2m} \frac{e^2}{c^2} \sum_{k, k', i} A(\vec{k}) A(\vec{k}') e^{i(\vec{k} + \vec{k}') \cdot \vec{x}_i} \quad (A.2)$$



This further is equal to

$$H_c = \frac{1}{2m} \frac{e^2}{c^2} \int \sum_{\vec{k}, \vec{k}', \vec{k}''} \delta n(\vec{k}'') a(\vec{k}) A(\vec{k}) e^{i(\vec{k} + \vec{k}' + \vec{k}'') \cdot \vec{x}_i} d\vec{x} \quad (A.3)$$

where  $\delta n(k'')$  is the perturbation in the electron density.

This is non zero is  $k'' = - (k+k')$ . If  $k'' \neq - (k+k')$ , then  $H_c$  remains the same as in (A.2). That is, it is not possible to eliminate the particle variables. Therefore transforming to second quantization formalism (A.2) becomes

$$H_c = \sum_{\vec{k}, \vec{k}', \vec{k}''} \frac{e^2}{2mc^2} \frac{\hbar}{2\mu} \frac{1}{\sqrt{\omega_k \omega_{k'}}} \left[ a_k^* a_{k'} + a_k^* a_{k'}^* \right] \times \\ C_{k+k'+k''}^* C_{k''} + H.C. \quad (A.4)$$

Thus one can see that  $H_c$  gives rise to the interactions mentioned above.



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